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MULTIPLE STONE-ČECH EXTENSIONS

(*DUAL STONE-ČECH EXTENSIONS*)

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ABSTRACT. For a nowhere (locally) compact space we iterate Stone-Čech compactification ω_1 many times to get a compact space where two or more disjoint dense subsets are C^*-embedded. The corresponding compact spaces we get for $\mathbb{Q}$ (the rationals), $\mathbb{P}$ (the irrationals) and $\mathbb{S}$ (the Sorgenfrey line) are not extremally disconnected, hence different from their absolutes.

1. INTRODUCTION

This talk originates from van Douwen's question in his paper "Remote points" (see §19 of [4]) that:
What happens if we repeat taking remainders of Stone-Čech compactifications of the rationals

$$\mathbb{Q}^* = \beta\,\mathbb{Q}\backslash\mathbb{Q},\ \ \mathbb{Q}^{**} = \beta\,\mathbb{Q}^*\backslash\mathbb{Q}^*,\ \ \mathbb{Q}^{***},\ \ \cdots.$$

He remarks that "it might be interesting to define $\mathbb{Q}^{(\alpha)}$'s, for $\alpha \geqslant \omega$, using inverse limits at limit stages" and that "there must be a γ for which the natural map from $\mathbb{Q}^{(\gamma+2)}$ to $\mathbb{Q}^{(\gamma)}$ is a homeomorphism." We will show in this paper that the least such γ is the first uncountable ordinal ω_1 (which we will denote by Ω for notational convenience).

Let K be a compact space of countable π-weight, partitioned as a disjoint union of two dense Lindelöf subspaces $K = K^- \cup K^+$. Then, in this paper, iterating Stone-Čech compactification $\omega_1 = \Omega$ many times, we will construct a compact space $\Omega(K) = K_\Omega^- \cup K_\Omega^+$ satisfying the following conditions:
(1) $\Omega(K)$ admits a perfect irreducible map $g : \Omega(K) \to K$ such that $g(K_\Omega^-) = K^-,\ \ g(K_\Omega^+) = K^+$.
(2) Both of K_Ω^-, K_Ω^+ are C^*-embedded in $\Omega(K)$.
Though, as is well known, the absolute (or the projective cover) of K also satisfies the corresponding conditions as above (1), (2), we can show, in most cases we deal with, that our compact space $\Omega(K)$ is not extremally disconnected, hence different from the absolute.

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Typical cases we are going to deal with are the following partitions.

Example 1. $K = [0,1], K^- = Q, K^+ = P$ where $Q = (0,1) \cap \mathbb{Q}$ and $P = [0,1] \backslash Q$. Obviously, Q is a homeomorphic copy of the rationals $\mathbb{Q}$, and P is that of the irrationals $\mathbb{P}$.

Example 2. $K=$ the Alexandroff double arrow space $\mathbb{A}$, i.e., the lexicographically ordered space $\mathbb{A} = [0,1] \times \{0,1\} \backslash \{(0,0),(1,1)\}$ which is the union of two dense sets $K^- = (0,1] \times \{0\}, K^+ = [0,1) \times \{1\}$, each of which is a copy of the Sorgenfrey line $\mathbb{S}$.

In this talk we show how to construct such an extension $\Omega(K)$ in general. The proofs and the details of its properties will appear in the forthcoming paper [6].

All spaces are assumed to be completely regular and Hausdorff, and maps are always continuous, unless otherwise stated. "Partition" is synonymous with "disjoint union."

As a suitable class for our purpose we consider the following class $\mathcal{L}$ consisting of Lindelöf spaces X such that

(i) X is *nowhere compact* (or nowhere locally compact), i.e., X has no compact neighborhood, and

(ii) every compact subset of X is included in some compact zero-set of X.

In terms of compactifications the condition (i) is equivalent to say that the remainder $cX \backslash X$ of any/some compactification cX of X is dense in cX, while the second one (ii) is equivalent to say that $cX \backslash X$ is Lindelöf for any/some compactification cX. The subclass of $\mathcal{L}$ consisting only of first countable spaces will be denoted by $\mathcal{L}(\text{1st})$.

The rationals $\mathbb{Q}$, the irrationals $\mathbb{P} = \mathbb{R} \backslash \mathbb{Q} \approx \omega^\omega$, the Sorgenfrey line $\mathbb{S}$ (i.e., the real line with the half-open interval topology) are the typical members of $\mathcal{L}(\text{1st})$. That $\mathbb{S}$ belongs to $\mathcal{L}(\text{1st})$ can be seen by regarding the double arrow space $\mathbb{A}$ in Example 2 as a compactification of $\mathbb{S}$. All of

$$\mathbb{P} \times \mathbb{Q}, \quad \mathbb{S} \times \mathbb{Q}, \quad \mathbb{Q} \times \mathbb{C}, \quad \mathbb{S} \times \mathbb{C} \quad \mathbb{S} \times \mathbb{P}$$

belong to $\mathcal{L}$. Note that $\mathbb{P} \times \mathbb{C}$ is nothing but $\mathbb{P}$ because

$$\mathbb{P} \times \mathbb{C} \approx \omega^\omega \times 2^\omega \approx (\omega \times 2)^\omega \approx \omega^\omega \approx \mathbb{P}.$$

For topological characterization of $\mathbb{P} \times \mathbb{Q}$ and $\mathbb{Q} \times \mathbb{C}$ see [7] and [8].

As a basic tool we use perfect irreducible maps, so we will list their properties needed here. Let g be a map from X onto Y. For a subset $U \subseteq X$ define $g^\circ(U) \subseteq Y$ by

$$y \in g^\circ(U) \text{ if and only if } g^{-1}(y) \subseteq U,$$

i.e., $g^\circ(U) = Y \backslash g(X \backslash U) \subseteq g(U)$. Note an obvious, but useful, formula

$$g^\circ(U \cap V) = g^\circ(U) \cap g^\circ(V)$$

for any sets $U, V \subseteq X$, which especially implies that $g^\circ(U) \cap g^\circ(V) = \emptyset$ whenever $U \cap V = \emptyset$. An onto map g is called *irreducible* if $g^\circ(U) \neq \emptyset$ for every non-empty open set U. A collection $\mathcal{B}$ of nonempty open sets of X is called a *π-base* for X if every nonempty open set in X contains some member of $\mathcal{B}$. The minimal cardinality of such a π-base is called the *π-weight* of X. Observe that any dense subspace of X has the same π-weight as X, and that any space of countable π-weight is separable. Consequently, any dense or open subset of a space of countable π-weight is also of countable π-weight, and hence separable. So, for example, all of $\mathbb{Q}$, $\beta\mathbb{Q}$, $\mathbb{Q}^* = \beta\mathbb{Q}\backslash\mathbb{Q}$ are of countable π-weight, and hence separable. A closed map with compact fibers are called *perfect.* We assume a perfect map is always onto.

Fact 1.1. (Properties of Closed Irreducible Maps)
Let $g : X \to Y$ be any closed irreducible map. Then
(1) *$g^\circ(U)$ is non-empty and open whenever U is. Moreover,*

$$\mathrm{cl}_Y\, g^\circ(U) = \mathrm{cl}_Y\, g(U) = g(\mathrm{cl}_X U)$$

for every open subset $U \subseteq X$, i.e., g carries a regular closed set $\mathrm{cl}_X U$ to a regular closed set $\mathrm{cl}_Y\, g^\circ(U)$.
(2) *g preserves ccc, i.e., X is ccc if and only if Y is. Similarly, g preserves density and π-weight. In case g is perfect irreducible, it also preserves nowhere compactness.*

Next lemma shows how we can produce perfect irreducible maps.

Lemma 1.2. *Let $\phi : X \to Y$ be a perfect map and let $\Phi : bX \to cY$ be its extension where bX and cY are some compactifications of X and Y respectively. Then Φ maps the remainder of X onto that of Y, i.e., $\Phi(bX\backslash X) = cY\backslash Y$. Moreover,*
(1) *ϕ is perfect irreducible if and only if Φ is.*
(2) *If ϕ is perfect irreducible and X (hence Y also) is nowhere compact, then the restriction of Φ to the remainders*

$$bX\backslash X \to cY\backslash Y$$

is also perfect irreducible. □

Perfect irreducible maps we encounter frequently in this paper are those induced by some homeomorphisms, i.e., when the above ϕ is an identity map.

For an open set U of X we can define its maximal open extension to βX by

$$\mathrm{Ex}(U) = \beta X \backslash \mathrm{cl}_{\beta X}(X\backslash U).$$

We denote the boundary of a subset W in Y by $\mathrm{Bd}_Y W$ so that $\mathrm{Bd}_Y W = \mathrm{cl}_Y W \backslash W$ if W is open in Y. Van Douwen [4] proved the following quite useful formula:

(1–0) $\qquad \mathrm{Bd}_{\beta X}\mathrm{Ex}(U) = \mathrm{cl}_{\beta X}\mathrm{Bd}_X(U)$ for every open set U in X.

A space with a clopen base is called 0-*dimensional*, and most spaces we deal with in this paper are 0-dimensional. As is well known (cf. 16.16 in [5]), for a Lindeöf space X the 0-dimensionality of X is equivalent with that of βX; in other words, the collection of $\mathrm{Ex}(U)$'s where U ranges over all clopen sets in X forms a clopen base for βX.

2. Construction of Dual Extensions

We use inverse systems only of the form

$$\{X_\xi,\, g_{\alpha,\beta},\, \xi\}$$

where ξ is an ordinal, and $g_{\alpha,\beta} : X_\beta \to X_\alpha$ $(\alpha < \beta < \xi)$ are bonding maps, and denote its inverse limit as $X_\xi = \varprojlim \{X_\alpha, g_{\alpha,\beta}, \xi\}$. Projections are denoted by $\pi_\alpha : X_\xi \to X_\alpha$, or $\pi_\alpha = \pi_\alpha^\xi = g_{\alpha,\xi}$. We assume all inverse systems in this paper are *continuous*, i.e.,

$$X_\eta = \varprojlim \{X_\alpha,\, g_{\alpha,\beta},\, \eta\}$$

for each limit $\eta < \xi$. Recall that, if we take a base $\mathcal{B}_\alpha$ for each X_α, the collection $\bigcup_{\alpha<\xi} \pi_\alpha^{-1}(\mathcal{B}_\alpha)$ forms a base for X_ξ.

The next lemma is well known for a system of compact spaces (cf. §11 in [1]); what we need here is for a system of Lindelöf spaces.

Lemma 2.1. (Factorization Lemma) *Suppose* $\mathrm{cof}(\xi) > \omega$, *and* $X_\xi = \varprojlim \{X_\alpha, g_{\alpha,\beta}, \xi\}$ *is Lindelöf. Then every map* $f : X_\xi \to \mathbb{R}$ *can be factorized as* $f = \widehat{f} \circ \pi_\alpha$ *for some* $\alpha < \xi$ *and some map* $\widehat{f} : X_\alpha \to \mathbb{R}$.

Proof. Let $\mathcal{B}$ be a countable open base of $\mathbb{R}$, and $f : X_\xi \to \mathbb{R}$. Take any $U \in \mathcal{B}$. Then, since $f^{-1}(U)$ is a cozero-set of X_ξ, it can be expressed that $f^{-1}(U) = \pi_{\alpha(U)}^{-1}(W)$ for some cozero-set W of $X_{\alpha(U)}$ with $\alpha(U) < \xi$. Put $\alpha = \sup\{\alpha(U) : U \in \mathcal{B}\} < \xi$. Then this α has the property that for every $U \in \mathcal{B}$ there exists an open set W of X_α such that $f^{-1}(U) = \pi_\alpha^{-1}(W)$. Therefore Lemma 2.1 follows from the next lemma. □

Lemma 2.2 (Yong [9]). *Let* $\pi : X \to Y$, $f : X \to Z$ *and suppose* π *is onto. Then* f *is factorized as* $f = \widehat{f} \circ \pi$ *for some map* $\widehat{f} : Y \to Z$ *if and only if the space* Z *has an open base* $\mathcal{B}$ *with the property that:*
For every $U \in \mathcal{B}$ *the open set* $f^{-1}(U)$ *takes the form* $f^{-1}(U) = \pi^{-1}(W)$ *for some open set* $W \subseteq Y$. □

Now let $K = X^{(0)} \cup X^{(1)}$ be a compact space with a partition into nowhere compact spaces $X^{(0)}, X^{(1)}$. Since both of $X^{(0)}, X^{(1)}$ are dense in K, we can see K as a compactification of either of $X^{(0)}$ or $X^{(1)}$. Put $X_0 = K$, $X_1 = \beta X^{(1)}$, $X^{(2)} = \beta X^{(1)} \backslash X^{(1)}$, and let

$$\Phi_0 : X_1 = \beta X^{(1)} = X^{(1)} \cup X^{(2)} \to X_0 = X^{(0)} \cup X^{(1)}$$

be the Stone extension of the identity map $id : X^{(1)} \to X^{(1)}$. Denote by

$$\phi_0 : X^{(2)} \to X^{(0)}$$

the restriction of Φ_0. Next, putting $X_2 = \beta X^{(2)}$, $X^{(3)} = \beta X^{(2)} \backslash X^{(2)}$, let

$$\Phi_1 : X_2 = \beta X^{(2)} = X^{(2)} \cup X^{(3)} \to X_1 = \beta X^{(1)} = X^{(1)} \cup X^{(2)}$$

be the Stone extension of the identity map $id : X^{(2)} \to X^{(2)}$. Denote by

$$\phi_1 : X^{(3)} \to X^{(1)}$$

the restriction of Φ_1. Repeating these procedures of Stone-Čech compactifications infinitely many times, we get mappings Φ_n, ϕ_n $(n \in \omega)$ such that

$$\Phi_n : X_{n+1} = X^{(n+1)} \cup X^{(n+2)} \to X_n = X^{(n)} \cup X^{(n+1)},$$

$$\text{where} \quad X_m = \beta X^{(m)}, \ \ X^{(m+1)} = \beta X^{(m)} \backslash X^{(m)} \ \text{ for } m \geqslant 1,$$

is the Stone extension of the identity map $id : X^{(n+1)} \to X^{(n+1)}$, and

$$\phi_n : X^{(n+2)} \to X^{(n)}$$

is the restriction of Φ_n. Then all of Φ_n, ϕ_n $(n \in \omega)$ are perfect irreducible. We can consider the system $\{X_n, \Phi_n\}_{n\in\omega}$ and its induced ones $\{X^{(2m)}, \phi_{2m+1}\}_{m\in\omega}$, $\{X^{(2m+1)}, \phi_{2m+2}\}_{m\in\omega}$ as inverse sequences, and take their limits

$$X_\omega = \varprojlim \{X_n, \Phi_n\}_{n\in\omega},$$

$$X_\omega^- = \varprojlim \{X^{(2m)}, \phi_{2m+1}\}_{m\in\omega}, \quad X_\omega^+ = \varprojlim \{X^{(2m+1)}, \phi_{2m+2}\}_{m\in\omega}.$$

Then it is easy to see that the projections $\pi_n^\omega : X_\omega \to X_n$ are perfect irreducible, and so, X_ω^-, X_ω^+ are nowhere compact and $X_\omega = X_\omega^- \cup X_\omega^+$ can be seen as a compactification of X_ω^-. Therefore, just replacing the starting $X_0 = X^{(0)} \cup X^{(1)}$ by $X_\omega = X_\omega^- \cup X_\omega^+$, we can repeat the Stone-Čech extensions as before to get $\{X_{\omega+n}, \Phi_{\omega+n}\}_{n\in\omega}$ and $X_{\omega+\omega} = \varprojlim \{X_{\omega+n}, \Phi_{\omega+n}\}_{n\in\omega}$. Let us do these extensions up to $\Omega = \omega_1$. (For notational simplicity we use Ω for the first uncountable ordinal ω_1.) Then we finally get a continuous inverse system of length Ω

$$X_\Omega = \varprojlim \{X_\alpha, \Phi_{\alpha,\beta}, \Omega\} \tag{2-0}$$

with the following properties:

(1) Each X_α $(\alpha \leqslant \Omega)$ is partitioned as $X_\alpha = X_\alpha^- \cup X_\alpha^+$ into two disjoint dense subsets, and

$X_\alpha^+ = X_{\alpha+1}^+$ for even α, while $X_\alpha^- = X_{\alpha+1}^-$ for odd α.
(An ordinal of the form $\gamma + 2m$ where γ is a limit ordinal and $m \in \omega$ is called "even," while an ordinal not even is "odd." Note that limit ordinals are even.)
(2) For any $\alpha < \beta < \Omega$ the bonding map $\Phi_{\alpha,\beta}$ is such that

$$\Phi_{\alpha,\beta} : X_\beta = X_\beta^- \cup X_\beta^+ \to X_\alpha = X_\alpha^- \cup X_\alpha^+$$

$$\Phi_{\alpha,\beta}(X_\beta^-) = X_\alpha^-, \quad \Phi_{\alpha,\beta}(X_\beta^+) = X_\alpha^+.$$

Moreover, $\Phi_{\alpha,\alpha+1}$ is the Stone extension of the following identity map: $id : X_{\alpha+1}^+ = X_\alpha^+$ for even α, and $id : X_{\alpha+1}^- = X_\alpha^-$ for odd α.

So, to be compatible with our beginning notation, we need to set

$$X_{2m}^+ = X_{2m+1}^+ = X^{(2m+1)}, \quad X_{2m+1}^- = X_{2m+2}^- = X^{(2m+2)}, \quad \Phi_{\alpha,\alpha+1} = \Phi_\alpha$$

for $m \in \omega$ and $\alpha < \omega+\omega$. In particular, $X_0 = X^{(0)} \cup X^{(1)} = X_0^- \cup X_0^+$, and we call any one of spaces X_0, X_0^-, X_0^+ the *starting space.*

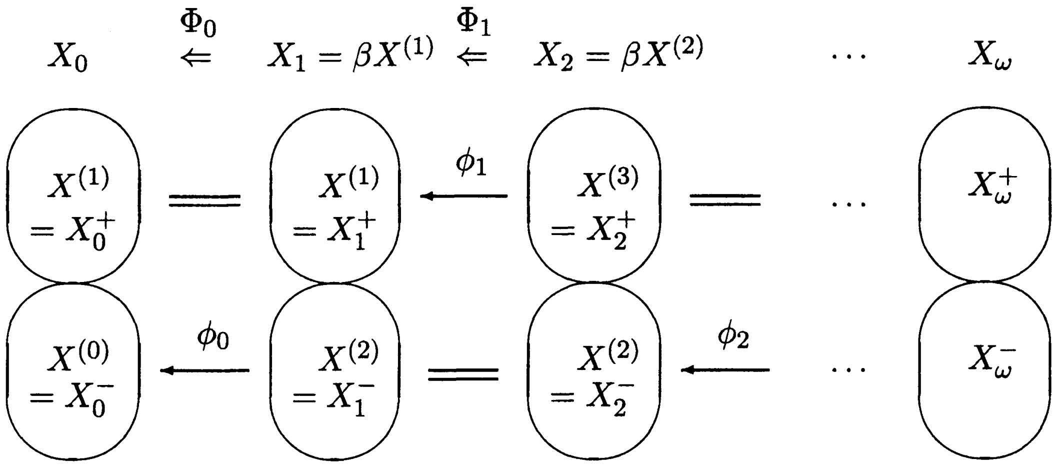

FIG. 1. The first ω steps

Naturally this inverse system $\{X_\alpha, \Phi_{\alpha,\beta}, \Omega\}$ has two subsystems

$$\{X_\alpha^-, \Phi_{\alpha,\beta}^-, \Omega\}, \quad \{X_\alpha^+, \Phi_{\alpha,\beta}^+, \Omega\}$$

with limits X_Ω^-, X_Ω^+ respectively, where

$$\Phi_{\alpha,\beta}^- : X_\beta^- \to X_\alpha^-, \quad \Phi_{\alpha,\beta}^+ : X_\beta^+ \to X_\alpha^+$$

are restrictions of $\Phi_{\alpha,\beta}$. The corresponding projections will be denoted by

$$\pi_\alpha : X_\Omega \to X_\alpha, \quad \pi_\alpha^- : X_\Omega^- \to X_\alpha^-, \quad \pi_\alpha^+ : X_\Omega^+ \to X_\alpha^+.$$

All maps $\Phi_{\alpha,\beta}$, $\Phi_{\alpha,\beta}^-$, $\Phi_{\alpha,\beta}^+$, π_α, π_α^-, π_α^+ are perfect irreducible. Consequently, if one of the beginning spaces X_0^-, X_0^+ belongs to the class $\mathcal{L}$, so do all of X_α^-, X_α^+ $(\alpha \leqslant \Omega)$. Note also that if one of X_0^-, X_0^+, X_0 has a countable

π-base, all of $X_\alpha^-, X_\alpha^+, X_\alpha$ $(\alpha \leqslant \Omega)$ have countable π-bases.

The factorization lemma implies

Theorem 2.3. (Dually C^*-embedded Extension)
Assume $X_0^- \in \mathcal{L}$, i.e., $X_0^+ \in \mathcal{L}$. Then X_Ω^-, $X_\Omega^+ \in \mathcal{L}$, and both of them are C^*-embedded in X_Ω, i.e., symbolically,

$$\beta(X_\Omega^-) = \beta(X_\Omega^+) = X_\Omega.$$

Proof. By symmetry it suffices to show that $X_\Omega^- = \varprojlim \{X_\alpha^-, \Phi_{\alpha,\beta}^-, \Omega\}$ is C^*-embedded in X_Ω. Let $f : X_\Omega^- \to [0,1]$ be any continuous function on X_Ω^-. Then, by the factorization lemma, we can find some $\alpha < \Omega$ and a continuous function $\widehat{f}$ on X_α^- such that $f = \widehat{f} \circ \pi_\alpha^-$. Once such an α is chosen, any $\beta > \alpha$ plays the same role as α. Therefore we can assume that α is odd. Then our construction assures that X_α^- is C^*-embedded in X_α, so that the bounded function $\widehat{f}$ can be extended to $h : X_\alpha \to [0,1]$. The function $h \circ \pi_\alpha : X_\Omega \to [0,1]$ is the desired extension of f. □

We call the space X_Ω in Theorem 2.3
the dual Stone-Čech Ω-extension of the partition $\mathcal{P} : X_0 = X_0^- \cup X_0^+$.
In general let $Y = Y^- \cup Y^+$ be a partition of a space Y into two dense subsets. Then we call $Y = Y^- \cup Y^+$ as a *dually C^*-embedded* partition of Y, if both of Y^-, Y^+ are C^*-embedded in Y. With this terminology Theorem 2.3 can be rephrased that
$X_\Omega = X_\Omega^- \cup X_\Omega^+$ is a dually C^*-embedded partition if $X_0^- \in \mathcal{L}$.

We can show that the space X_Ω of (2-0) depends only on the partition $\mathcal{P}$, so that in particular we get the same space $X_\Omega = \Omega(X_0)$ if we exchange the role of X_0^- and X_0^+ in the above construction. For the proof of this fact see the forthcoming paper [6]. So, let us denote X_Ω by $\Omega(\mathcal{P})$, or simply by $\Omega(X_0)$ when the partition $\mathcal{P}$ is clear.

Now suppose a nowhere compact space $X \in \mathcal{L}$ is given. Then, regarding $X = X_0^-$, we get the subspace X_Ω^- of X_Ω which is uniquely determined by the given space X. Let us denote this X_Ω^- by $\Omega(X)$. Then Theorem 2.3 implies

$$\Omega(\beta X) = \beta\left(\Omega(X)\right)$$

for $X \in \mathcal{L}$. For example, we have

$$\Omega([0,1]) = \Omega(\beta\mathbb{Q}) = \beta\left(\Omega(\mathbb{Q})\right) = \Omega(\beta\mathbb{P}) = \beta\left(\Omega(\mathbb{P})\right)$$

for the partition of $[0,1]$ in Example 1, and

$$\Omega(\mathbb{A}) = \Omega(\beta\mathbb{S}) = \beta\left(\Omega(\mathbb{S})\right)$$

for the partition of $\mathbb{A}$ in Example 2. We can show that $\Omega(\mathbb{A})$ is not homeomorphic with $\Omega([0,1])$, by proving that $\Omega(\mathbb{A})$ contains no dense set of first category which is C^*-embedded (see [6]).

Note that our construction becomes trivial if the given partition $X_0 = X_0^- \cup X_0^+$ itself is dually C^*-embedded. Fortunately we can prove that is not the case if $X_0^- \in \mathcal{L}(1\text{st})$, i.e.,

Theorem 2.4. ([6]) *Assume $X^{(0)} = X_0^- \in \mathcal{L}(1\text{st})$. Then no bonding map*

$$\Phi_{\alpha,\beta} : X_\beta \to X_\alpha \quad (\alpha < \beta < \Omega)$$

is one to one.

3. Common Boundary Points

Let S be a dense subset of T. A point $p \in T \backslash S$ is called *remote from S*, or a *remote point w.r.t.* (S,T), if $p \notin \mathrm{cl}_T F$ for every nowhere dense closed subset F of S. In case $T = \beta S$ we simply call such a point p as a *remote point of* S. Van Douwen [3, 4], and independently Chae and Smith [2], have shown that:

Fact 3.1. *Every non-pseudocompact space of countable π-weight has $2^{\mathfrak{c}}$ many remote points.*

A space T is said to be *extremally disconnected at a point $p \in T$* (see [4]) if $p \notin \mathrm{cl}_T\, U_1 \cap \mathrm{cl}_T\, U_2$ for every pair of disjoint open sets U_1, U_2 in T. We call such a point p an *extremally disconnected point* of T, or simply, an *e.d.* point of T. Obviously a space T is extremally disconnected if every point of T is an e.d. point. If S is dense in T, we always have $\mathrm{cl}_T U = \mathrm{cl}_T(U \cap S)$ for every open set U of T. So, an equivalent definition of an e.d. point is given using only open subsets of any dense subset $S \subseteq T$:

$p \in T$ is an e.d. point if and only if $p \notin \mathrm{cl}_T\, V_1 \cap \mathrm{cl}_T\, V_2$ for every pair of disjoint open sets V_1, V_2 in S.

Note that this definition does not depend on the choice of the dense subset S, while it is clear that the notion of remote points depends on the choice of the dense subset S. Note also that in case T, S are ccc (e.g., of countable π-weight), we can choose the above U_1, U_2 as cozero-sets of T, and V_1, V_2 as cozero-sets of S. The next fact proved by van Douwen [4] tells that "remote" implies "e.d." implies "C^*-embedded."

Fact 3.2. (1) *If $p \in \beta X \backslash X$ is remote from X, then p is an e.d. point of βX.* (2) *Let X be dense in Y, and $p \in Y \backslash X$. If p is an e.d. point of Y, then X is C^*-embedded in $X \cup \{p\}$ $(\subseteq Y)$.*

The proof of the above (1) uses the formula (1-0) in §1.

Let us call a non-e.d. point of T as a "common boundary point" of T, that is, $p \in T$ is a *common boundary* point of T if $p \in \mathrm{cl}_T\, U_1 \cap \mathrm{cl}_T\, U_2$ for some pair of disjoint open sets U_1, U_2 in T. Similarly, a closed subset $A \subseteq T$

is called a *common boundary* set in T if $A \subseteq \mathrm{cl}_T U_1 \cap \mathrm{cl}_T U_2$ for some pair of disjoint open sets U_1, U_2 in T. Let us abbreviate "common boundary" to "co-boundary." (Such p, A are called "2-point" or "2-set" in [4]. We prefer geometric terminology.) Let $\mathrm{Ed}(T)$ denote the set of all e.d. points of T, and put $\mathrm{Cob}(T) = T \backslash \mathrm{Ed}(T)$ which is the set of all co-boundary points of T.

Theorem 3.3. ([6]) *Assume $X_0^-, X_0^+ \in \mathcal{L}$ and that the starting space $X_0 = X_0^- \cup X_0^+$ contains a compact co-boundary set F_0 such that $F_0^- = F_0 \cap X_0^-$, $F_0^+ = F_0 \cap X_0^+$ are nowhere compact and $F_0 \subseteq \mathrm{cl}\, U_0 \cap \mathrm{cl}\, V_0$ in X_0 for some disjoint open sets U_0, V_0 in X_0. Then we can find a compact co-boundary set F_Ω in $X_\Omega = \Omega(X_0)$ such that*

$$\pi_0(F_\Omega) = F_0 \quad \textit{and} \quad F_\Omega \subseteq \mathrm{cl}_{X_\Omega}(U_\Omega) \cap \mathrm{cl}_{X_\Omega}(V_\Omega)$$

for disjoint open sets $U_\Omega = \pi_0^{-1}(U_0)$, $V_\Omega = \pi_0^{-1}(V_0)$ in $X_\Omega = \Omega(X_0)$. Hence, for each $x \in F_0$ we get

$$\pi_0^{-1}(x) \cap \mathrm{Cob}(X_\Omega) \neq \emptyset.$$

Consequently, $\mathrm{Cob}(X_\Omega) = X_\Omega \backslash \mathrm{Ed}(X_\Omega)$ *is not empty, i.e., $X_\Omega = \Omega(X_0)$ is not extremally disconnected.*

Next easy lemma tells when the hypothesis of Theorem 3.3 is satisfied.

Lemma 3.4. *Suppose $Y \in \mathcal{L}(\mathrm{1st})$, and that Y contains a nowhere dense closed subset $F \in \mathcal{L}(\mathrm{1st})$. Then we can find disjoint open subsets U, V such that $F \subseteq \mathrm{cl}\, U \cap \mathrm{cl}\, V$ in Y.* $\square$

From this lemma it is easy to see that the typical examples $\mathbb{Q}, \mathbb{P}, \mathbb{S} \in \mathcal{L}(\mathrm{1st})$ satisfy the hypothesis of Theorem 3.3. Let us illustrate a specific simple partition of $\mathbb{Q}$, as in Lemma 3.4, into the form $U \cup F \cup V$ where $F = \mathrm{cl}\, U \cap \mathrm{cl}\, V$, using the standard Cantor set. Consider the standard middle-thirds Cantor set

$$\mathbb{C} = [0,1] \backslash \bigcup_{n \in \omega} (a_n, b_n)$$

where (a_n, b_n) $(n \in \omega)$ are disjoint open intervals in $(0,1)$ with end points $a_n, b_n \in \mathbb{Q}$. Choose $c_n \in (a_n, b_n) \cap \mathbb{P}$ for each $n \in \omega$ and put

$$U = Q \cap \bigcup_{n \in \omega} (a_n, c_n), \quad V = Q \cap \bigcup_{n \in \omega} (c_n, b_n), \quad F = Q \cap \mathbb{C}.$$

Then Q is partitioned as $Q = U \cup F \cup V$, and $F = \mathrm{cl}_Q U \backslash U = \mathrm{cl}_Q V \backslash V \approx Q$ is nowhere dense closed in Q.

We can conclude from Theorem 3.3 and Lemma 3.4 that neither $\Omega([0,1])$ nor $\Omega(\mathbb{A})$ is extremally disconnected.

4. Generalization to Multiple Extensions

Now let us consider more general partitions. Suppose a compact space K has a partition $\mathcal{P}$ such that

$$(4\text{–}0) \qquad \mathcal{P}: \quad K = (\bigcup_{i \in A} L^i) \cup S$$

where $A \subseteq \omega$, $2 \leqslant |A| \leqslant \omega$, and each L^i $(i \in A)$ is dense in K. We put no particular condition on $S = K \setminus \bigcup_{i \in A} L^i$; for example, S need not be dense, or it may happen $S = \emptyset$. The case of §2 is

$$L^0 = X^-, \; L^1 = X^+, \; A = \{0, 1\}, \; S = \emptyset.$$

Using inverse limits similar to §2, we can construct

$$(4\text{–}1) \qquad \Omega(\mathcal{P}) = (\bigcup_{i \in A} L^i_\Omega) \cup S_\Omega,$$

where $L^i_\Omega = \pi^{-1}(L^i)$, $S_\Omega = \pi^{-1}(S)$, and $\pi : \Omega(\mathcal{P}) \to K$ is a perfect irreducible projection, with the following property similar to Theorem 2.3.

Theorem 4.1. ([6]) *Suppose a partition $\mathcal{P}$ of* (4-0) *is such that each dense subset L^i $(i \in A)$ is Lindeöf. Then the corresponding Lindeöf dense subset L^i_Ω in* (4-1) *is C^*-embedded in $\Omega(\mathcal{P})$, i.e., $\Omega(\mathcal{P}) = \beta(L^i_\Omega)$ for each $i \in A$.*

In view of this theorem we can call $\Omega(\mathcal{P})$
the multiple Stone-Čech Ω-extension w.r.t. the dense sets L^i $(i \in A)$ of the partition $\mathcal{P}$.
We may think of various partitions $\mathcal{P}$, and accordingly various multiple extensions. See [6] for further details.

5. Conclusion

As is well known, for every space X there exists an extremely disconnected space $\mathbf{E}(X)$ called the "absolute," with a perfect irreducible map onto X. Our space $\Omega(X)$ lies in between X and $\mathbf{E}(X)$, and will serve as a useful device to mediate X and $\mathbf{E}(X)$.

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